

Generalized Dyck paths of bounded height

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Abstract

Generalized Dyck paths (or discrete excursions) are one-dimensional paths that take their steps in a given finite set S , start and end at height 0, and remain at a non-negative height. Bousquet-Mélou showed that the generating function E_k of excursions of height at most k is of the form F_k/F_{k+1} , where the F_k are polynomials satisfying a linear recurrence relation. We give a combinatorial interpretation of the polynomials F_k and of their recurrence relation using a transfer matrix method. We then extend our method to enumerate discrete meanders (or paths that start at 0 and remain at a non-negative height, but may end anywhere). Finally, we study the particular case where the set S is symmetric and show that several simplifications occur.

1 Introduction and notations

A *Dyck path* is a one-dimensional path taking its steps in $\{-1, 1\}$, starting and ending at 0 and visiting only non-negative points (Figure 1, left). It is well-known that the number of Dyck paths of length $2n$ is the n th *Catalan number* C_n [10, Chapter 6]:

$$C_n = \frac{1}{n+1} \binom{2n}{n}.$$

Moreover, let $E = E(t)$ be the generating function of Dyck paths. This generating function satisfies the following algebraic equation:

$$1 - E + t^2 E^2 = 0.$$

A *generalized Dyck path* (or *discrete excursion*) takes its steps in a given finite set S instead of $\{-1, 1\}$. A *discrete meander* is a slightly more general path: it takes its steps in S , starts at 0 and visits only non-negative points, but may end anywhere (figure 1, right).

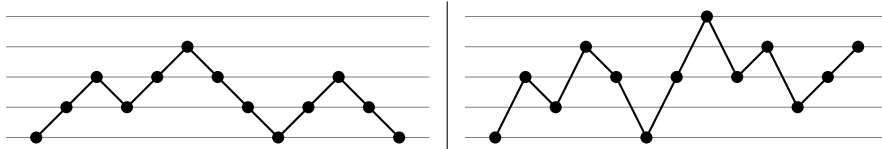


Figure 1: Left: a Dyck path of height 3. Right: a discrete meander with set of steps $S = \{\pm 1, \pm 2\}$, of height 4 and with final height 3.

A large number of papers enumerate generalized Dyck paths with varying amounts of generality. Methods used include some properties of Laurent series [6], grammars [7, 8, 5, 1], or the kernel method [2, 3]. Assume that all steps s in S have a weight ω_s taken in some field of characteristic 0 (typically a field of fractions with one or several variables over \mathbb{Q}). Let E and M be the generating functions of excursions and meanders, respectively, according to the weights ω_s . Banderier and Flajolet [2] showed that both these generating functions are algebraic. More precisely, let $a = \max S$ and $b = -\min S$; one may compute polynomials of degree $\binom{a+b}{a}$ canceling E and M .

In this paper, we consider excursions and meanders with *bounded height*, that is, that never go above a certain level, say k . We denote by E_k the generating function of excursions of height at most k and by $E_{k,\ell}$ the generating function of meanders of height at most k with final height ℓ . We also denote by M_k the generating function of meanders of height at most k regardless of final height.

Bousquet-Mélou [3] proved that the generating function E_k is of the form:

$$E_k = \frac{F_k}{F_{k+1}},$$

where the F_k are polynomials in the weights ω_s . She also proved, with the use of symmetric functions, that the polynomials F_k satisfy a linear recurrence relation of order $\binom{a+b}{a}$. In other words, we have

$$\sum_{k \geq 0} F_k z^k = \frac{N(z)}{D(z)},$$

where the polynomial $D(z)$ has degree $\binom{a+b}{a}$ (the polynomial $N(z)$ has degree $\binom{a+b}{a} - a - b$). Moreover, it can be seen that the polynomial $D(z)$ cancels the generating function of excursions E .

To our knowledge, the only cases of the generating function $E_{k,\ell}$ that were studied before with general steps are $\ell = 0$, that corresponds to excursions, and $\ell = k$, that corresponds to *culminating paths* [4]. In this paper, we use a transfer matrix method to compute both generating functions E_k and $E_{k,\ell}$.

We now set some notations. As argued in [3], excursions of height at most k are walks in a finite graph, with vertices $\{0, \dots, k\}$ and an arc from i to j if $j - i$ is in S . We denote by A_k the adjacency matrix of this graph. If s is in \mathbb{Z} , let β_s be the quantity:

$$\beta_s = \delta_{s,0} - \begin{cases} \omega_s & \text{if } s \in S, \\ 0 & \text{otherwise.} \end{cases}$$

In this way, the (i, j) entry of the matrix $1 - A_k$ is β_{j-i} .

If m and n are integers, we use the notation $\llbracket m, n \rrbracket$ to denote the set of integers i such that $m \leq i \leq n$. We also use the notation $\mathbb{N}_{\geq n}$ to denote the set of integers greater than or equal to n . Finally, if X is a set and n an integer, we call n -subset of X a subset of X of cardinality n .

The paper is organized as follows. Section 2 deals with bounded excursions, re-proving Bousquet-Mélou's results using a simple transfer matrix method. This method is expanded in Section 3 to cover bounded meanders as well. In Section 4, we discuss the case where the set S is symmetric and show that several simplifications occur.

2 Bounded excursions

Our first step to enumerate excursions of height at most k is the same as in [3]. As these excursions are walks that go from 0 to 0 in the graph described by the matrix A_k , the generating function E_k is the $(0, 0)$ entry of the matrix $(1 - A_k)^{-1}$ (see [9, Chapter 4]). Therefore, we have:

$$E_k = \frac{F_k}{F_{k+1}}, \quad (1)$$

where F_k is the determinant of the matrix $1 - A_{k-1}$, with the convention $F_0 = 1$.

As the entry (i, j) of the matrix $1 - A_{k-1}$ is β_{j-i} , we have, by the definition of the determinant:

$$F_k = \sum_{\sigma \in \mathfrak{S}_k} \varepsilon(\sigma) \prod_{i=0}^{k-1} \beta_{\sigma(i)-i}.$$

In the following, we use this expression to compute F_k . The permutations σ that it involves are, of course, bijections from $\llbracket 0, k-1 \rrbracket$ to itself; however, we find it more convenient to regard them as bijections from \mathbb{N} to itself that fix all points in $\mathbb{N}_{\geq k}$.

Definition 1. Let I be an a -subset of the set $\llbracket -b, a-1 \rrbracket$. We call *I-permutation* of order k a bijection from \mathbb{N} to $I \cup \mathbb{N}_{\geq a}$ that fixes all points in $\mathbb{N}_{\geq k}$.

Note that for a I -permutation to exist when $k < a$, all points in $\llbracket k, a-1 \rrbracket$ must be in the set I since they are fixed points. Also note that a $\llbracket 0, a-1 \rrbracket$ -permutation of order k is the same as a standard permutation of order k ; for this reason, we set $I_0 = \llbracket 0, a-1 \rrbracket$.

We denote by $\mathfrak{S}_k^{(I)}$ the set of I -permutations of order k . Let σ be a I -permutation. We define the number of *inversions* of σ , denoted by $\text{inv}(\sigma)$, in the same manner as a regular permutation:

$$\text{inv}(\sigma) = \#\{(i, j) \mid i < j \text{ and } \sigma(i) > \sigma(j)\}.$$

We define the *signature* of σ , denoted by $\varepsilon(\sigma)$, to be the number $(-1)^{\text{inv}(\sigma)}$. We also define the quantity $\beta(\sigma)$ to be:

$$\beta(\sigma) = \varepsilon(\sigma) \prod_{i=0}^{k-1} \beta_{\sigma(i)-i}.$$

Finally, we call *head* of σ the value $\sigma(0)$; we call *tail* of σ the mapping τ defined for all $n \in \mathbb{N}$ by:

$$\tau(n) = \sigma(n+1) - 1.$$

We now denote by \mathbf{F}_k the vector indexed by the a -subsets of $\llbracket -b, a-1 \rrbracket$ and the entries of which are:

$$\mathbf{F}_k[I] = \sum_{\sigma \in \mathfrak{S}_k^{(I)}} \beta(\sigma).$$

The above remark means that the entry I_0 of this vector coincides with F_k .

We compute the vector \mathbf{F}_k using a simple transfer matrix method, in a manner similar to [9, Example 4.7.7 and Proposition 4.7.8a]. If I is a subset of

$\llbracket -b, a-1 \rrbracket$ and s an integer, we denote by $\varepsilon_s(I)$ the number -1 to the power of the number of elements of I lower than s :

$$\varepsilon_j(I) = (-1)^{\#\{i \in I \mid i < s\}}.$$

We denote by \mathbf{T} the matrix whose rows and columns are indexed by the a -subsets of $\llbracket -b, a-1 \rrbracket$ and the entries of which are:

$$\mathbf{T}[I, J] = \begin{cases} \varepsilon_s(I)\beta_s & \text{if } I \cup \{a\} = (J+1) \cup \{s\}, \\ 0 & \text{otherwise.} \end{cases} \quad (2)$$

Figure 2 shows two instances of this matrix. The first corresponds to the paths with set of steps $S = \{0, \pm 1\}$ (Motzkin paths); the second corresponds to $S = \{0, \pm 1, \pm 2\}$ (this is known as the *basketball* problem, and studied in [1]). In both cases, we prefer to represent the graph \mathcal{G} , the vertices of which are the a -subsets of $\llbracket -b, a-1 \rrbracket$ and the adjacency matrix of which is \mathbf{T} .

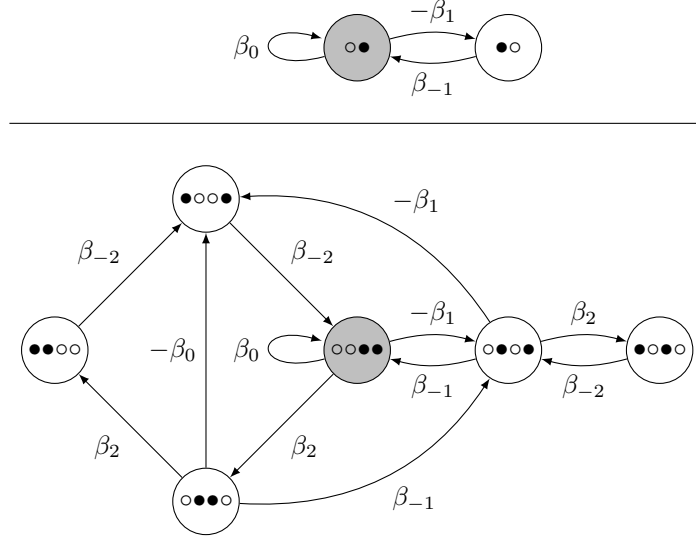


Figure 2: The graph \mathcal{G} for the sets of steps $S = \{0, \pm 1\}$ (above) and $S = \{0, \pm 1, \pm 2\}$ (below). In each case, the subset I of $\llbracket -b, a-1 \rrbracket$ corresponding to each vertex is represented by a sequence of \bullet and \circ (e.g. $\circ\bullet\bullet\bullet$ corresponds to the subset $\{-1, 1\}$ of $\llbracket -2, 1 \rrbracket$). The vertex corresponding to the set I_0 is colored gray.

To help the reader familiarise himself with this definition, we start by stating two basic properties of the matrix \mathbf{T} . We make use of them later.

Lemma 2. *Let I and J be a -subsets of $\llbracket -b, a-1 \rrbracket$ and assume that $\mathbf{T}[I, J] \neq 0$. The two following implications hold.*

1. *If $-b$ is in I , then J is such that $I \cup \{a\} = (J+1) \cup \{-b\}$. In this case, we have $\mathbf{T}[I, J] = \beta_{-b}$.*
2. *If $a-1$ is not in J , then I is equal to $J+1$. In this case, we have $\mathbf{T}[I, J] = (-1)^a \beta_a$.*

We can check this lemma by looking at Figure 2: all vertices with a label starting with a \bullet have only one outgoing arc, with label β_{-b} ; all vertices with a label ending with a \circ have only one ingoing arc, with label $(-1)^a \beta_a$.

Proof. By the definition of the matrix \mathbf{T} , if I and J are such that $\mathbf{T}[I, J] \neq 0$, we have $I \cup \{a\} = (J + 1) \cup \{s\}$ for some s in $\llbracket -b, a - 1 \rrbracket$.

Let us prove the first implication. As J is a subset of $\llbracket -b, a - 1 \rrbracket$, the point $-b$ cannot be in $J + 1$. Therefore, if $-b$ is in I , we have $s = -b$. To prove the second implication, we note that the point a is obviously in $I \cup \{a\}$ and therefore in $(J + 1) \cup \{s\}$. Therefore, if $a - 1$ is not in J , we have $s = a$. \square

We now define some particular vertices of the graph \mathcal{G} , defined as cyclic permutations of I_0 . Specifically, if m is such that $-b \leq m \leq a$, let I_m be the vertex of \mathcal{G} defined as the set $\llbracket m, m + a - 1 \rrbracket$ modulo $a + b$, where the values modulo $a + b$ are taken in $\llbracket -b, a - 1 \rrbracket$. Since $I_{-b} = I_a$, these vertices number $a + b$.

Lemma 3. *If $0 < m \leq a$, any walk in the graph \mathcal{G} going from I_m visits I_0 . If $-b \leq m < 0$, any walk in the graph \mathcal{G} going backwards from I_m visits I_0 .*

Proof. The first result stems from the fact that $-b$ is in I_m if $0 < m \leq a$. Lemma 2 entails that the only arc of \mathcal{G} going from I_m goes to I_{m-1} . Thus, any walk going from I_m reaches I_0 in m steps.

Likewise, the second result stems from the fact that $a - 1$ is not in I_m if $-b \leq m < 0$. Lemma 2 entails that the only arc of \mathcal{G} going to I_m goes from I_{m+1} . This means that any walk going backwards from I_m reaches I_0 . \square

Proposition 4. *The vectors \mathbf{F}_k satisfy for $k \geq 0$:*

$$\mathbf{F}_{k+1} = \mathbf{T}\mathbf{F}_k.$$

Proof. To prove the proposition, we show that for every a -subset I of $\llbracket -b, a - 1 \rrbracket$, we have:

$$\mathbf{F}_{k+1}[I] = \sum_J \mathbf{T}[I, J] \mathbf{F}_k[J].$$

To do this, we let σ be a I -permutation of order $k + 1$ such that $\beta(\sigma) \neq 0$. Let $s = \sigma(0)$ be the head of σ and τ be its tail, defined above. Since the number of inversions created by 0 in the permutation σ is equal to the number of elements of I lower than s , we have:

$$\beta(\sigma) = \varepsilon_s(I) \beta_s \beta(\tau).$$

It remains to show that τ is a J -permutation, where $I \cup \{a\} = (J + 1) \cup \{s\}$. Since $\beta_s \neq 0$, we have $s \in \llbracket -b, a \rrbracket$. Thus, we may write:

$$\sigma: \mathbb{N} \rightarrow \{s\} \cup (I \cup \{a\} \setminus \{s\}) \cup \mathbb{N}_{\geq a+1}.$$

From this, we deduce:

$$\tau: \mathbb{N} \rightarrow J \cup \mathbb{N}_{\geq a}.$$

From its definition, the set J is *a priori* an a -subset of $\llbracket -b - 1, a - 1 \rrbracket$; however, since $\beta(\tau) \neq 0$, the point $-b - 1$ cannot be in J . This finishes the proof. \square

From Proposition 4, we derive the following theorem, which already appears in [3] but is found using completely different methods.

Theorem 5. *The generating function of the polynomials F_k is:*

$$\sum_{k \geq 0} F_k z^k = \frac{N(z)}{D(z)},$$

where $D(z)$ is the determinant of $1 - z\mathbf{T}$ and $N(z)$ is the (I_0, I_0) cofactor of the same matrix.

Moreover, the degree of the polynomial $D(z)$ is $\binom{a+b}{a}$; the degree of $N(z)$ is $\binom{a+b}{a} - a - b$.

Proof. The only possible I -permutation of order 0 is the identity, which is an I_0 -permutation. This implies that the polynomial $F_0^{(I)}$ is 1 if $I = I_0$ and 0 otherwise. With Proposition 4, we deduce that the polynomial F_k is equal to the entry (I_0, I_0) in the matrix \mathbf{T}^k .

The generating function $\sum_{k \geq 0} F_k z^k$ is therefore equal to the entry (I_0, I_0) in the matrix $(1 - z\mathbf{T})^{-1}$. The announced form follows from Cramer's rule.

To compute the degree of the polynomial $D(z)$, we let $d = \binom{a+b}{a}$. Since \mathbf{T} is a $d \times d$ matrix, the polynomial $D(z)$ has degree at most d ; the coefficient of z^d in this polynomial is (up to a sign) $\det(\mathbf{T})$. Denoting by \mathfrak{S}_d the set of permutations of the set of a -subsets of $\llbracket -b, a-1 \rrbracket$, we have:

$$\det(\mathbf{T}) = \sum_{\pi \in \mathfrak{S}_d} \varepsilon(\pi) \prod_I \mathbf{T}[I, \pi(I)].$$

Let π be a permutation with a nonzero contribution in this sum. Lemma 2 asserts that:

1. if $-b$ is in I , then $I \cup \{a\} = (\pi(I) + 1) \cup \{-b\}$;
2. if $a-1$ is not in $\pi(I)$, then $I = \pi(I) + 1$.

Condition 1 determines $\pi(I)$ if $-b$ is in I ; Condition 2 determines $\pi(I)$ if $-b$ is not in I . The permutation π is thus uniquely determined. Moreover, the values of $\mathbf{T}[I, \pi(I)]$ are given by Lemma 2. This gives, up to a sign, the value of $\det(\mathbf{T})$:

$$\det(\mathbf{T}) = \pm \beta_{-b} \binom{a+b-1}{a-1} \beta_a \binom{a+b-1}{a},$$

which is nonzero. Therefore, the polynomial $D(z)$ has degree d .

To compute the degree of $N(z)$, we adopt a more combinatorial point of view: we regard the determinant $D(z)$ as the generating function of *configurations of cycles* of the graph \mathcal{G} , counted up to a sign, where z accounts for the number of vertices visited by the configuration. Let π_0 be the unique permutation, defined above, that contributes to the dominant term of $D(z)$; the permutation π_0 can be interpreted as the only configuration of cycles that visits all vertices of \mathcal{G} .

The cofactor $N(z)$ is the generating function of the configurations of cycles that avoid the vertex I_0 . By Lemma 3, such a configuration cannot visit any of the vertices I_m (since if it would, it would also visit I_0). This means that $N(z)$ has degree at most $d - a - b$. Moreover, we easily check that the vertices I_m form a cycle of the configuration π_0 ; by removing this cycle, we thus obtain a configuration of cycles visiting $d - a - b$ vertices and not visiting I_0 . Therefore, the degree of $N(z)$ is exactly $d - a - b$. \square

As examples, let us compute the polynomials $D(z)$ and $N(z)$ corresponding to the two examples of Figure 2. In the case of Motzkin paths $S = \{0, \pm 1\}$, let us take the weights $\omega_0 = 0$ and $\omega_1 = \omega_{-1} = t$ (*i.e.*, the case of standard Dyck paths). Theorem 5 shows that the polynomials F_k satisfy:

$$\sum_{k \geq 0} F_k z^k = \frac{1}{1 - z + t^2 z^2},$$

which is equivalent the recurrence relation:

$$\begin{aligned} F_0 &= F_1 = 1, \\ F_k &= F_{k-1} - t^2 F_{k-2} \quad \text{if } k \geq 2. \end{aligned}$$

These polynomials are commonly known as the *Fibonacci polynomials*, due to the similarities with the recurrence relation of the Fibonacci numbers.

In the basketball case $S = \{0, \pm 1, \pm 2\}$, let us take the weights $\omega_0 = 0$, $\omega_1 = \omega_{-1} = t_1$ and $\omega_2 = \omega_{-2} = t_2$. In this case, the polynomials $D(z)$ and $N(z)$ are:

$$\begin{aligned} D(z) &= (1 + t_2 z)^2 (1 - z - 2t_2 z + t_1^2 z^2 + 2t_2 z^2 + 2t_2^2 z^2 - t_2^2 z^3 - 2t_2^3 z^3 + t_2^4 z^4); \\ N(z) &= (1 + t_2 z)(1 - t_2 z). \end{aligned}$$

As we can see, a simplification by a factor of $1 + t_2 z$ occurs in the computation of the fraction $N(z)/D(z)$. This implies that the polynomials F_k follow a linear recurrence relation of order 5 instead of 6, as shown in [1]. The factorisation of the polynomial $D(z)$ is also predicted in [3]; it is linked to the fact that the set S and the weights β_s are *symmetric*.

3 Bounded meanders

We enumerate bounded meanders in the same manner as bounded excursions. Specifically, the generating function $E_{k,\ell}$ counts walks from 0 to ℓ in the graph described by the matrix A_k . Therefore, we have

$$E_{k,\ell} = \frac{F_{k,\ell}}{F_{k+1}}, \quad (3)$$

where $F_{k,\ell}$ is the $(\ell, 0)$ cofactor of the matrix $1 - A_k$. Obviously, we have $F_{k,0} = F_k$.

We again compute the cofactors $F_{k,\ell}$ using a transfer matrix method. We start by writing $F_{k,\ell}$ in terms of the determinant of the matrix $1 - A_k$ with the ℓ th row and the 0th column cut. This reads:

$$\begin{aligned} F_{k,\ell} &= (-1)^\ell \det \left(\begin{pmatrix} \beta_{j-i+1} & \text{if } i < \ell \\ \beta_{j-i} & \text{if } i \geq \ell \end{pmatrix} \right)_{0 \leq i, j \leq k-1} \\ &= (-1)^\ell \sum_{\sigma \in \mathfrak{S}_k} \varepsilon(\sigma) \prod_{i=0}^{\ell-1} \beta_{\sigma(i)-i+1} \prod_{i=\ell}^{k-1} \beta_{\sigma(i)-i}. \end{aligned}$$

For every integer s , we set $\tilde{\beta}_s = -\beta_{s+1}$. We also define, for every permutation σ of order k , the quantity:

$$\beta_\ell(\sigma) = \varepsilon(\sigma) \prod_{i=0}^{\ell-1} \tilde{\beta}_{i-\sigma(i)} \prod_{i=\ell}^{k-1} \beta_{i-\sigma(i)}.$$

We rewrite the above formula as:

$$F_{k,\ell} = \sum_{\sigma \in \mathfrak{S}_k} \beta_\ell(\sigma).$$

Let now I be an $a-1$ -subset of the set $\llbracket -b-1, a-2 \rrbracket$. Let $\tilde{\mathfrak{S}}_k^{(I)}$ be the set of I -permutations of order k with respect to the set $\tilde{S} = S-1$. We define the following polynomial:

$$F_{k,\ell}^{(I)} = \sum_{\sigma \in \tilde{\mathfrak{S}}_k^{(I)}} \beta_\ell(\sigma).$$

We also denote by $\mathbf{F}_{k,\ell}$ the vector whose I -component is $F_{k,\ell}^{(I)}$.

Let $\tilde{\mathbf{T}}$ be the transfer matrix defined in Section 2 corresponding to the set \tilde{S} and the weights $\tilde{\beta}_s$. Let \mathbf{U} be the matrix whose rows are indexed by the $a-1$ -subsets of $\llbracket -b-1, a-2 \rrbracket$, whose columns are indexed by the a -subsets of $\llbracket -b, a-1 \rrbracket$ and whose entries are:

$$\mathbf{U}[I, J] = \begin{cases} 1 & \text{if } I \cup \{a-1\} = J, \\ 0 & \text{otherwise.} \end{cases} \quad (4)$$

We also let $\tilde{\mathcal{G}}$ be the graph with adjacency matrix $\tilde{\mathbf{T}}$. Let \mathcal{H} be the graph $\mathcal{G} \cup \tilde{\mathcal{G}}$, with additional arcs between the vertices of \mathcal{G} and $\tilde{\mathcal{G}}$ coded by the matrix \mathbf{U} . Examples are shown in Figure 3.

Proposition 6. *The vectors $\mathbf{F}_{k,\ell}$ satisfy for $k \geq \ell \geq 0$:*

$$\begin{aligned} \mathbf{F}_{k+1,\ell+1} &= \tilde{\mathbf{T}} \mathbf{F}_{k,\ell}; \\ \mathbf{F}_{k,0} &= \mathbf{U} \mathbf{F}_k. \end{aligned}$$

Proof. The proof of the first identity follows the same lines as that of Proposition 4. Let I be an $a-1$ -subset of $\llbracket -b-1, a-2 \rrbracket$ and σ be in $\tilde{\mathfrak{S}}_{k+1}^{(I)}$. Let s be the head and τ be the tail of σ . We have:

$$\beta_{\ell+1}(\sigma) = \varepsilon_s(I) \tilde{\beta}_s \beta_\ell(\tau).$$

The rest of the proof is identical to that of Proposition 4.

To prove the second identity, we let I be an $a-1$ -subset of $\llbracket -b-1, a-2 \rrbracket$ and σ be in $\tilde{\mathfrak{S}}_k^{(I)}$. By definition, we have:

$$\beta_0(\sigma) = \beta(\sigma).$$

Moreover, assume that $\beta_0(\sigma) = \beta(\sigma) \neq 0$. This means that $-b-1$ cannot be in I . Therefore, the set $J = I \cup \{a-1\}$ is an a -subset of $\llbracket -b, a-1 \rrbracket$. Again by definition, the mapping σ is in $\mathfrak{S}_k^{(J)}$. This completes the proof. \square

Theorem 7. *The bivariate generating function of the polynomials $F_{k,\ell}$ satisfies:*

$$\sum_{k \geq \ell \geq 0} F_{k,\ell} u^\ell z^k = \sum_{k \geq 0} F_k(u) z^k = \frac{\tilde{N}(u, z)}{\tilde{D}(uz)D(z)},$$

where $D(z)$ is the determinant of $1 - z\mathbf{T}$, $\tilde{D}(z)$ is the determinant of $1 - z\tilde{\mathbf{T}}$, and $\tilde{N}(u, z)$ may be computed as:

$$\tilde{N}(u, z) = \sum_{\mathbf{U}[I, J]=1} \text{Cof}[I, \tilde{I}_0](1 - uz\tilde{\mathbf{T}}) \text{Cof}[I_0, J](1 - z\mathbf{T}).$$

The polynomial $D(z)$ has degree $\binom{a+b}{a}$, the polynomial $\tilde{D}(z)$ has degree $\binom{a+b}{a-1}$, and the polynomial $\tilde{N}(u, z)$ has a dominant term in

$$z^{\binom{a+b+1}{a}} - a - b - 1 u^{\binom{a+b}{a-1}} - a.$$

Proof. As the only I_0 -permutation of order 0 is the identity, Proposition 6 entails that the polynomial $F_{k,\ell}$ is equal to the entry (\tilde{I}_0, I_0) in the matrix $\tilde{\mathbf{T}}^\ell \mathbf{U} \mathbf{T}^{k-\ell}$. In other terms, $F_{k,\ell}$ is the generating function of walks from \tilde{I}_0 to I_0 in the graph \mathcal{H} taking ℓ steps in the graph \mathcal{G} and $k - \ell$ steps in the graph \mathcal{G} . The generating function $\sum_{k,\ell} F_{k,\ell} u^\ell z^k$ is thus equal to the entry (\tilde{I}_0, I_0) in the matrix $(1 - wz\tilde{\mathbf{T}})^{-1} \mathbf{U} (1 - z\mathbf{T})^{-1}$. This yields the announced form.

Let us now compute the degrees; let $\tilde{d} = \binom{a+b}{a-1}$. Theorem 5 shows that the polynomial $\tilde{D}(z)$ has degree \tilde{d} .

To compute the dominant term of the polynomial $N(u, z)$, we first remark that if I and J are such that $\mathbf{U}[I, J] = 1$, we have $-b-1 \notin I$ and $a-1 \in J$. This implies that Lemma 2 is still valid in the graph \mathcal{H} . Let \tilde{I}_m , for $-b-1 \leq m \leq a-1$, be the vertices of $\tilde{\mathcal{G}}$ defined in the same way as the vertices I_m . Since Lemma 2 is valid, Lemma 3 is also valid regarding both the vertices I_m and \tilde{I}_m .

We now consider the polynomial $N(u, z)$. This polynomial is the generating function of configurations in the graph \mathcal{H} composed of elementary cycles and a self-avoiding walk going from \tilde{I}_0 to I_0 ; the variable u takes into account the number of arcs of $\tilde{\mathcal{G}}$ in the configuration. By Lemma 3, such a configuration cannot visit a vertex \tilde{I}_m with $0 \leq m \leq a-1$ (since it would contain an arc going into \tilde{I}_0), nor can it visit a vertex I_m with $-b \leq m < 0$ (since it would contain an arc going from I_0). This proves that the dominant term is at most in

$$z^{\tilde{d}-a+d-b-1} u^{\tilde{d}-a}.$$

Let $\tilde{\pi}_0$ and π_0 be the only configurations of cycles visiting all vertices of the graphs $\tilde{\mathcal{G}}$ and \mathcal{G} , respectively. Consider the configuration consisting of:

- all cycles of $\tilde{\pi}_0$ except the one containing the vertices \tilde{I}_m ;
- all cycles of π_0 except the one containing the vertices I_m ;
- the self-avoiding walk $\tilde{I}_0 \rightarrow \cdots \rightarrow \tilde{I}_{-b} \rightarrow I_a \rightarrow \cdots \rightarrow I_0$.

We check that this configuration contains $\tilde{d} - a$ arcs of \mathcal{G} and $d - b - 1$ arcs of \mathcal{G} . We thus derive the dominant term of the polynomial $N(u, z)$. \square

We now consider the generating function M_k of all meanders of height at most k regardless of final height. From (3), we find

$$M_k = \frac{F_k(1)}{F_{k+1}}, \quad (5)$$

where the polynomial G_k is the sum of $F_{k,\ell}$ for all ℓ between 0 and k . The generating function of the polynomials G_k is found by setting $w = 1$ in the expression of Theorem 7; this proves that the polynomials G_k follow a linear recurrence relation of order $\binom{a+b}{a} + \binom{a+b}{a-1} = \binom{a+b+1}{a}$.

Let us now take the two examples detailed in Section 2. The case of Dyck paths (Figure 3, left) is very simple since the graph $\tilde{\mathcal{G}}$ has only one vertex. If we set $\omega_0 = 0$ and $\omega_1 = \omega_{-1} = t$, the generating function of the polynomials $F_{k,\ell}$ is:

$$\sum_{k \geq \ell \geq 0} F_{k,\ell} u^\ell z^k = \frac{1}{(1-tuz)(1-z+t^2z^2)}.$$

In other words, we have:

$$F_{k,\ell} = t^\ell F_{k-\ell}.$$

Let us now examine the case where $S = \{0, \pm 1, \pm 2\}$, $\omega_0 = 0$, $\omega_1 = \omega_{-1} = t_1$ and $\omega_2 = \omega_{-2} = t_2$ (Figure 3, right). The polynomials $\tilde{D}(z)$ and $\tilde{N}(u, z)$ are:

$$\begin{aligned} \tilde{D}(z) &= 1 - t_1 z - t_2 z^2 - t_1 t_2^2 z^3 + t_2^4 z^4; \\ \tilde{N}(u, z) &= (1 + t_2 z)(1 - t_2 z + t_1 t_2 u z^2 - t_2^3 u^2 z^3 + t_2^4 u^2 z^4). \end{aligned}$$

Once again, a simplification by a factor of $1 + t_2 z$ occurs in the computation of the generating function $\sum_{k,\ell} F_{k,\ell} u^\ell z^k$.

4 Symmetric set of steps

We now consider the special case where the set of steps S is *symmetric*, that is, where $-S = S$ and $\omega_{-i} = \omega_i$ for all $i \in S$. Bousquet-Mélou already considers this case and shows [3] that the generating function $E(t)$ is canceled by a polynomial of degree 2^a instead of $\binom{2a}{a}$.

While we were not able to recover Bousquet-Mélou's results with our methods, we show that another phenomenon occurs: namely, the polynomials F_k factor into two parts, and simplifications occur in the computation of the generating functions of meanders $E_{k,\ell}$ and M_k . These simplifications are basically due to the fact that as S is symmetric, the entry (i, j) of the matrix A_k is identical to the entry $(k-i, k-j)$. Define the following two matrices:

$$\begin{aligned} A_k^+ &= \left(\omega_{j-i} + \begin{cases} \omega_{k-j-i} & \text{if } j < k/2 \\ 0 & \text{if } j = k/2 \end{cases} \right)_{0 \leq i, j \leq k/2}, \\ A_k^- &= \left(\omega_{j-i} - \omega_{k-j-i} \right)_{0 \leq i, j < k/2} \end{aligned}$$

(all values of ω_s are taken to be 0 if s is not in S). If k is an odd number, both matrices have the same dimension and the condition $j = k/2$ never occurs; if k is an even number, the dimension of A_{k+1}^+ is one more than that of A_{k+1}^- . In

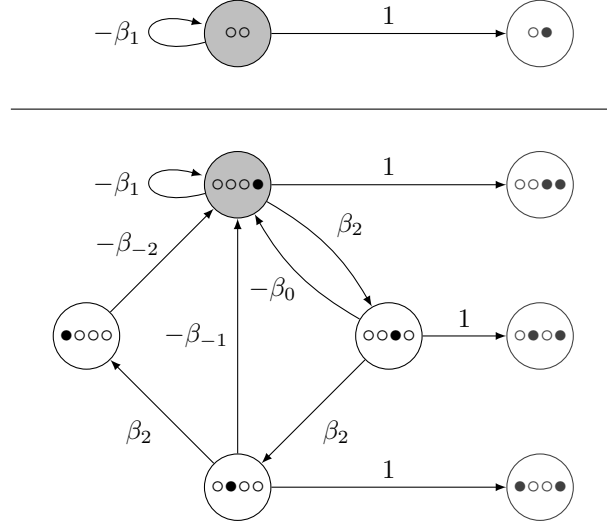


Figure 3: The graphs $\tilde{\mathcal{G}}$ for the sets of steps $S = \{0, \pm 1\}$ (above) and $S = \{0, \pm 1, \pm 2\}$ (below). Graphical conventions are identical to those of Figure 2, with the vertex corresponding to \tilde{I}_0 colored gray. Arcs with weight 1, coded by the matrix \mathbf{U} and leading to vertices of the graph \mathcal{G} shown in Figure 2 are also shown.

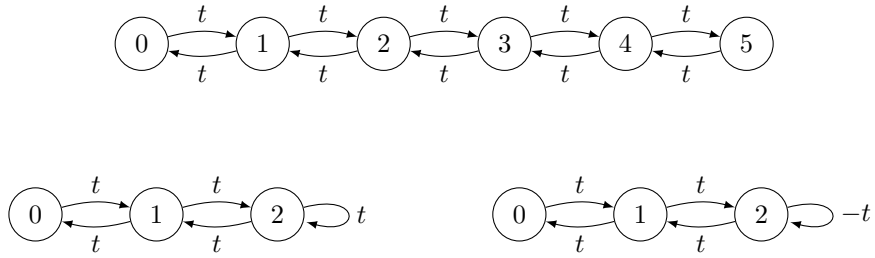


Figure 4: The graphs corresponding to the three matrices A_3 , A_3^+ and A_3^- with $S = \{\pm 1\}$ and $\omega_1 = \omega_{-1} = t$. They differ only by the vertex 3.

both cases, the sum of the two dimensions is $k + 1$. The graphs with adjacency matrices A_k^+ and A_k^- are illustrated in Figure 4.

We denote by F_k^+ and F_k^- the determinants of the matrices $1 - A_{k-1}^+$ and $1 - A_{k-1}^-$, respectively. We also denote by $F_{k,\ell}^+$ and $F_{k,\ell}^-$ the $(\ell, 0)$ cofactors of the matrices $1 - A_k^+$ and $1 - A_k^-$, respectively.

Theorem 8. *The polynomial F_k satisfies for all integers k :*

$$F_k = F_k^+ F_k^-.$$

Moreover, the generating functions $E_{k,\ell}$ satisfy, for all ℓ such that $0 \leq \ell < k/2$:

$$E_{k,\ell} + E_{k,k-\ell} = \frac{F_{k,\ell}^+}{F_{k+1}^+}; \quad E_{k,\ell} - E_{k,k-\ell} = \frac{F_{k,\ell}^-}{F_{k+1}^-}.$$

Finally, if k is even and $\ell = k/2$, we have:

$$E_{k,\ell} = \frac{F_{k,\ell}^+}{F_{k+1}^+}.$$

Proof. Let $\mathcal{B} = (b_0, \dots, b_k)$ be the canonical basis of the underlying vector space of the matrix A_k . We denote by \mathcal{B}_0 , \mathcal{B}_1 and \mathcal{B}_2 the following collections of vectors:

$$\begin{aligned} \mathcal{B}_0 &= (b_i)_{0 \leq i < k/2}; \\ \mathcal{B}_1 &= (b_{k/2}) \text{ if } k \text{ is even and } \emptyset \text{ otherwise}; \\ \mathcal{B}_2 &= (b_{k-i})_{0 \leq i < k/2}. \end{aligned}$$

Since the entry (i, j) of the matrix A_k is equal to the entry $(k-i, k-j)$, the matrix A_k written as a block matrix with respect to the basis $(\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2)$ looks like:

$$A_k = \left(\begin{array}{c|c|c} B & U & C \\ \hline V & x & V \\ \hline C & U & B \end{array} \right).$$

Let P be the following passage matrix:

$$P = \left(\begin{array}{c|c|c} 1 & 0 & 1 \\ \hline 0 & 1 & 0 \\ \hline 1 & 0 & -1 \end{array} \right),$$

where 1 denotes the identity matrix of the appropriate dimension. We compute:

$$P^{-1}A_kP = \left(\begin{array}{c|c|c} B+C & U & 0 \\ \hline 2V & x & 0 \\ \hline 0 & 0 & B-C \end{array} \right) = \left(\begin{array}{c|c} A_k^+ & 0 \\ \hline 0 & A_k^- \end{array} \right).$$

All identities of the theorem are readily derived from this form. \square

This result shows that the generating functions $E_{k,\ell}$ can be computed using smaller polynomials than expected, since the matrices A_k^+ and A_k^- (and thus their determinants and cofactors) are about twice as small as A_k . Moreover, we can derive from the proposition a simplified expression for the generating function of meanders M_k . By writing M_k as the sum of $E_{k,\ell}$ for all $0 \leq \ell \leq k$ and grouping the terms by pairs, we find:

$$M_k = \frac{F_k^+(1)}{F_{k+1}^+}. \quad (6)$$

This expression involves smaller polynomials than (5) (a simplification occurs by a factor of F_{k+1}^-).

Theorem 9. *The generating functions of the polynomials F_k^+ and F_k^- are of the form:*

$$\sum_{k \geq 0} F_k^+ z^k = \frac{N^+(z)}{D(z^2)}, \quad \sum_{k \geq 0} F_k^- z^k = \frac{N^-(z)}{D(z^2)},$$

where $D(z)$ is the polynomial defined in Theorem 5 and $N^+(z)$ and $N^-(z)$ are polynomials. Moreover, the bivariate generating functions of the polynomials $F_{k,\ell}^+$ and $F_{k,\ell}^-$ are of the form:

$$\sum_{k \geq \ell \geq 0} F_{k,\ell}^+ u^\ell z^k = \frac{\tilde{N}^+(u, z)}{\tilde{D}(uz^2)D(z^2)}; \quad \sum_{k \geq \ell \geq 0} F_{k,\ell}^- u^\ell z^k = \frac{\tilde{N}^-(u, z)}{\tilde{D}(uz^2)D(z^2)},$$

where $\tilde{D}(z)$ is the polynomial defined in Theorem 7 and $\tilde{N}^+(u, z)$ and $\tilde{N}^-(u, z)$ are polynomials.

Proof. This theorem is a consequence of a fact hinted at in Figure 4: the matrices A_k^+ and A_k^- are nearly identical to the matrices A_{k^+} and A_{k^-} , respectively, where $k^+ = \lfloor \frac{k}{2} \rfloor$ and $k^- = \lfloor \frac{k-1}{2} \rfloor$. More precisely, the term ω_{k-j-i} is zero if $k-j-i > a$; since $j \leq k/2$, this is true whenever $i < k/2 - a$.

For simplicity, we only prove the results associated to the matrix A_k^- , but the case of A_k^+ is identical. The proof follows the same techniques used in the previous sections. If I is a a -subset of $[-a, a-1]$, we denote by \mathbf{F}_{k+1}^- the vector whose I -coefficient is:

$$\mathbf{F}_k^-[I] = \sum_{\sigma \in \mathfrak{S}_{k^-}^{(I)}} \varepsilon(\sigma) \prod_{i=0}^{k^- - 1} (\beta_{\sigma(i)-i} - \omega_{k-1-\sigma(i)-i}).$$

Assume now that $k-1 > 2a$ and examine the first term of the product: the above remark entails that $\omega_{k-\sigma(i)-i} = 0$, which means that the first term is equal to $\beta_{\sigma(i)-i}$. This allows us to repeat the proof of Proposition 4. As the matrix A_k^- minus its first row and first column is equal to A_{k-2}^- , we find:

$$\mathbf{F}_k^- = \mathbf{T}\mathbf{F}_{k-2}^-.$$

In the same way, we define the vectors $\mathbf{F}_{k,\ell}^-$; by repeating the proof of Proposition 6, we find, if k is sufficiently large:

$$\begin{aligned} \mathbf{F}_{k+2,\ell+2}^- &= \tilde{\mathbf{T}}\mathbf{F}_{k,\ell}^-; \\ \mathbf{F}_{k,0}^- &= \mathbf{U}\mathbf{F}_k^-. \end{aligned}$$

All the identities of the theorem are derived from these recurrence relations. \square

As an example, we take the Fibonacci polynomials, corresponding to the set $S = \{\pm 1\}$ (see Section 2). The values of the polynomials F_k^+ and F_k^- are given by:

$$\begin{aligned} F_{2k}^+ &= F_k - tF_{k-1}; & F_{2k+1}^+ &= F_{k+1} - t^2F_{k-1}; \\ F_{2k}^- &= F_k + tF_{k-1}; & F_{2k+1}^- &= F_k. \end{aligned}$$

Obviously, these four sequences of polynomials follow the same recurrence relation as the polynomials F_k .

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